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Optimization of Fins Under Wet Conditions Using Variational Principle

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A variational principle is adopted to determine the optimum profiles of three commonly used fins, namely, longitudinal, annular, and spine operating in dehumidifying conditions. The optimization analysis is carried out to determine the shape of the fin by satisfying the maximization of heat transfer duty for a design constraint selected as either fin volume or both fin volume and length. The heat transfer analysis of the three common types of fins is presented through a unified formulation. From the optimization study, comparison of results between dry, fully wet, and partially wet surface fins has also been furnished. From the present result, it can be highlighted that the optimum condition of wet surface fins differs from that of dry surface fins due to the condensation of moisture taking place on the fin surface, and this disparity in results increases considerably with the increase in relative humidity due to evolving more latent heat of condensation. Unlike the dry surface condition, the parameters which are mainly responsible for the wet fin surface at an optimum point are of psychometric properties, viz. relative humidity, ambient and base temperatures, and design constraints associated with the optimization process. Finally, it may be noted that the design variables of a wet fin at an optimum condition also depend upon the aforementioned parameters.

Nomenclature

a	=	constant determined from the conditions of humid air at the fin base and fin tip
b	=	slop of a saturation line in the psychometric chart, K^{-1}
C	=	nondimensional integration constant used in Eq. (16)
C_p	=	specific heat of humid air, $J\,kg\,K^{-1}$
F	=	functional defined in Eq. (11)
h	=	convective heat transfer coefficient, $W\,m^{-2}\,K^{-1}$
h_m	=	mass transfer coefficient, $kg\,m^{-2}\,S^{-1}$
h_{fg}	=	latent heat of condensation, $J\,kg^{-1}$
k	=	thermal conductivity of the fin material, $W\,m^{-1}\,K^{-1}$
L	=	dimensionless fin length, hl/k
L_0	=	dimensionless wet length in partially wet fins, hl_0/k
L_k	=	dimensionless parameter defined in Eq. (22l)
Le	=	Lewis number, see Eq. (3)
l	=	fin length, m
l_0	=	wet length in partially wet fins, m
p, s	=	constants, see Eq. (2)
Q	=	dimensionless heat transfer rate, see Eq. (9).
q	=	heat transfer rate through a fin, W
R_i	=	dimensionless base radius, hr_i/k
r_i	=	base radius for annular fins, m
T	=	temperature, K
U	=	dimensionless fin volume, see Eq. (10)
V	=	fin volume (volume per unit width for longitudinal fins), m^3
X, Y	=	dimensionless coordinates, hx/k and hy/k, respectively
x, y	=	coordinates, m
Y_0	=	dimensionless thickness, hy_0/k
y_0	=	semithickness of a fin at which dry and wet parts are separated, m
Z_1, Z_2	=	dimensionless parameters defined in Eqs. (21g) and (21h), respectively
α	=	parameter defined in Eqs. (22c), (22f), and (22j)
θ	=	dimensionless fin temperature, $(T_a - T)/(T_a - T_b)$
θ_p	=	dimensionless temperature parameter, see Eq. (6)
λ	=	Lagrange multiplier
ξ	=	notation defined in Eq. (6)
ϕ	=	dimensionless temperature, $\theta + \theta_p$
ϕ_0	=	dimensionless temperature, $1 + \theta_p$
ω	=	specific humidity of air, kg of water vapor per kg of dry air

Subscripts

a	=	ambient
b	=	base
d	=	dew point
max	=	maximum
opt	=	optimum
t	=	tip

I. Introduction

FINS or extended surfaces are frequently used in heat exchangers for effectively improving the overall heat transfer performance. The simple design of fins and their stability for different combined heat and mass transfer conditions have made them a popular augmentation device. The fin-and-tube heat exchangers are employed in conventional air conditioning systems for air cooling and dehumidifying. In the evaporator of air conditioning equipment, the surface temperature of the fin is generally below the dew point of the surrounding air. As a result, moisture is condensed on the fin surface to evolve latent heat. Thus, in this application, mass transfer occurs simultaneously with the heat transfer. Depending upon the fin base, fin tip, and dew point temperatures of the surrounding air, fin surface may be classified as dry, partially wet, and fully wet conditions. Thermal performance of different surface conditions of a fin depends on the fin profile and thermophysical and psychometric properties of air. However, the augmentation of heat transfer is associated with the increase of volume, weight, and cost of the heat exchanger equipment because of the addition of fins. Therefore, there is a continuous endeavor by researchers to determine the optimum shape of a fin after satisfying either the maximization of heat transfer rate for a particular fin volume or the minimization of fin volume for a given heat transfer duty. In general, two different approaches are considered for the optimization of any fin design problem. Through a

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rigorous technique, the profile of a fin for a particular geometry (flat or curved primary surface) may be obtained such that the criteria of the maximum heat transfer for a given fin volume or minimum fin volume for a given heat transfer duty is satisfied. In a parallel activity, the optimum dimensions of a fin of a given profile (rectangular, triangular, etc.) are determined from the solution of the optimality criteria. Although the resulting profile obtained from the first case of optimum design is superior in respect to heat transfer rate per unit volume, it may be limited to use in actual practice because the resulting profile shape would be difficult to manufacture and fabricate. However, in practice, such theoretical shape would first be calculated and then a triangular profile approximating the base two-thirds of the fin would be used [1]. Such a triangular fin transfers heat per unit weight, which is closer to that analytical optimum value. Thus, many researchers [2–11] have already engaged to determine the optimum fin shape.

Schmidt [2] was the first researcher to forward a systematic approach for the optimum design of fins under a convective environmental condition. By a heuristic argument, he proposed that for an optimum shape of a cooling fin, the temperature should be a linear function with the fin length. Later, Duffin [3] exhibited rigorous proof on the optimality criteria of Schmidt [2] through the calculus of variation. Liu [4] extended the variational principle to find out the optimum profile of fins with internal heat generation. The optimization of radiating fins was addressed by Liu [5] and Wilkins [6]. The performance of annular fins of different profiles subject to locally variable heat transfer coefficients has been investigated by Mokheimer [7]. Now, it can be mentioned from the optimization study that the past works [2–7] were done with the consideration of "length of arc idealization" (LAI).

Maday [8] eliminated LAI and obtained the optimum profile through a numerical integration. It is of interest to note that, with the improvement suggested by Maday [8], an optimum convecting fin neither has a linear temperature profile nor possesses a concave parabolic shape. The profile shape contains a number of ripples and was denoted a "wavy fin." The same exercise was repeated for radial fins by Guceri and Maday [9]. Later, Razelos and Imre [10] applied Pontryagin's minimum principle to find out minimum mass of convective fins with variable heat transfer coefficients. Hati and Rao [11] determined the optimum profile of a one-dimensional fin under a convective–radiative condition by using classical control theory. To find the optimum dimensions of a spine subjected to a variable heat transfer coefficient, Natarajan and Shenoy [12] adopted a variational principle. The optimum dimensions of convecting–radiating fins have been determined for several fin shapes by Razelos and Kakatsios [13]. The optimal shape profiles for cooling fins of high and low conductivity has been determined by Bobaru and Rachakonda [14] with the consideration of a two-dimensional cross section of a periodic array of fins. Hanin and Campo [15] forecasted a shape of a straight cooling fin of minimum volume, taking into account the LAI. From the result, they have highlighted that the volume of the optimum circular fin found in their work is 6.21–8 times smaller than the volume of the corresponding Schmidt's parabolic optimum fin. Arauzo et al. [16] addressed an elementary analytic procedure for the quick estimation of the heat transfer characteristics of annular fins of hyperbolic profile. A generalized methodology for the optimum design of thin fins with uniform volumetric heat generation was described by Kundu and Das [17].

Leon et al. [18] studied the effect of the pressure drop caused by flow resistance of a heat sink on the cooling fin shape. The aerodynamic shape fin was established by satisfying the maximum heat transfer flux as well as the minimum flow resistance. Fabbri [19] proposed a genetic algorithm for fin profile optimization to determine the optimum profile shape under a convective environment. He obtained an optimum profile as a different polynomial order. Fabbri [20] investigated the heat transfer performance of optimized dissipaters with longitudinal fins of asymmetrical cross section and compared that with optimized dissipaters with symmetrical fins. He studied a problem of optimizing the shape and the spacing of the fins of a thermal dissipater cooled by a fluid in laminar flow by assigning two different polynomial lateral profiles to the fin.

There are plenty of practical applications in which extended surface heat transfer is involved in two-phase flow conditions. For example, when humid air encounters a cold surface of cooling coils whose temperature is maintained below the dew point temperature, condensation of moisture will take place, and mass and heat transfer occur simultaneously. Many investigations [21–35] have been devoted to analyze the effect of condensation on the performance of different geometric fins. It is noteworthy to mention that, for each instance, a suitable fin geometry has been selected a priori to make the analysis.

For the combined heat and mass transfer, the mathematical formulation becomes complex to determine the overall performance analysis of a wet fin. Based on the modified dry fin formula, Threlkeld [21] and McQuiston [22] determined the one-dimensional fin efficiency of a rectangular longitudinal fin for a fully wet surface condition. Using a quasi-linear one-dimensional model, Kilic and Onat [23] demonstrated the performance analysis of vertical rectangular wet fins with the consideration of constant heat and mass transfer coefficients along the fin. Toner et al. [24] have chosen the same technique to analyze the rectangular and triangular fins when condensation of moisture occurs on the fin surface. Coney et al. [25] numerically investigated the fin performance of vertical rectangular fins with condensation from humid air. During calculation of the mass transfer under dehumidification of air over a rectangular fin, taken into account was the thermal resistance of the condensate film and the use of a second degree polynomial to relate the humidity ratio with the dry bulb temperature.

The analysis of one-dimensional fin assembly heat transfer with dehumidification has been done numerically by Kazeminejad [26]. He has incorporated the ratio of sensible to total heat transfer into the analysis of fin assemblies during the dehumidification process. Kazeminejad et al. [27] determined numerically the performance of a cooling and dehumidifying vertical rectangular fin with variable heat transfer coefficients.

An analytical solution for the efficiency of a longitudinal straight fin under dry, fully wet, and partially wet surface conditions was introduced elaborately by Wu and Bong [28] first with considering temperature and humidity ratio differences as the driving forces for heat and mass transfer process. For the establishment of an analytical solution, a linear relationship between the humidity ratio and the corresponding saturation temperature of air was taken. Later, lot of analytical works [29–32] on the performance and optimization analysis of wet fins was carried out by applying this linear relationship. A technique to determine the performance and optimization of straight tapered longitudinal fins subject to simultaneous heat and mass transfer has been established analytically by Kundu [29]. Sala El-Din [30] developed an analytical model to predict the thermal performance of the fully and partially wet fin assembly of rectangular fins. The performance and optimum dimensions of a new fin, namely, step reduction in cross section (SRC) profile, subject to simultaneous heat and mass transfer, have been investigated by Kundu [31]. In his work, a comparative study has also been made between rectangular and SRC profile fins when they are operated in wet conditions. Hong and Web [32] calculated the fin efficiency for wet and dry circular fins with a constant thickness.

The analysis of two-dimensional heat conduction in fins subject to simultaneous heat and mass transfer has been developed by many investigators for a specified geometry. Kazeminejad et al. [33] demonstrated numerically the effect of dehumidification of air on the performance of eccentric annular fins. A two-dimensional fin efficiency analysis of combined heat and mass transfer in elliptic fins has been established numerically by Lin and Jang [34]. They have considered a quadratic variation of humidity ratio with the saturation temperature for calculating the latent heat. A new reduction method for analyzing the heat and mass transfer characteristics of wavy fin-and-tube heat exchangers under dehumidifying conditions has been proposed by Pirompugd et al. [35].

From the thorough literature survey summarized in the preceding paragraphs, the author found that all the previous investigations had been focused on determination of the optimum profile subjected to convective, radiative, or convective–radiative conditions. However, in refrigeration, air conditioning, and many practical applications, fin surface temperature is below the dew point of the surrounding humid air and, as a result, mass transfer takes place simultaneously with the heat transfer process. Thorough research works have already been devoted to analyzing the performance and optimization of wet fins and, to carry out their analyses, a suitable fin geometry has been chosen a priori in every investigation. Therefore, no research activity has been reported until now to determine the optimum profile of a fin under dehumidifying conditions. However, the optimum profile fin may be employed in air conditioning apparatus, especially in aircraft where reduction of weight is always given an important priority. This reason has been motivated by the analysis of the present study.

In this work, the problem of thin fins under a dehumidifying condition of practical interest is formulated with the treatment by a calculus of variation. Euler's equations are obtained by formal variational methods. A general mathematical theory has been developed for obtaining the required fin shape of three common types of fins, namely, longitudinal, spine, and annular fins, by satisfying maximizing heat transfer duty for a given fin volume, or both fin volume and length. The preset analysis is formulated for the dry, partially, and fully wet surface conditions. The effect of wet conditions over fin surfaces on the optimum profile shape and its dimensions has also been studied. From the results, it can be highlighted that, whether a surface becomes dry, partially, or fully wet at an optimum point, the main deciding factor is air relative humidity.

II. Physical Problem and Mathematical Formulations

Geometrical configurations of three types of thin fins and the coordinate system from which the conservation equations are derived are shown in Fig. 1. In refrigeration and air conditioning applications, heat is transferred from the surrounding air to the refrigerant via fins. Depending upon the fin base, fin tip, and dew point temperatures, fin surfaces may be classified into three categories. If the temperature of the entire fin surface is lower than the dew point of the surrounding air, there occurs both sensible and latent heat transferred from the air to the fin, and so the fin is fully wet. The fin is partially wet if the fin base temperature is below the dew point, while the fin tip temperature is above the dew point of the surrounding air. If the temperature of the entire fin surface is higher than the dew point, only sensible heat is transferred, and so the fin is fully dry. On the other hand, depending upon applications and design criteria, the shape of fins has many possibilities. For circular primary surface, circumferential fins are employed. The longitudinal and pin fins are generally used for the flat primary surface.

In the present analysis, it is assumed that the condensate thermal resistance to heat flow is negligibly small because the condensate film is much thinner than the boundary layer in the dehumidification process. Under such circumstances, it follows that the heat transfer coefficient is not influenced significantly by the presence of condensation. The condensation takes place when fin surface temperature is below the dew point of the surrounding air and, for the calculation, specific humidity of the saturated air on the wet surface is assumed to be a linear function with the local fin temperature. This assumption is due to the smaller temperature range involved in practical applications between fin base and dew point temperatures and, within this small range, the saturation curve on the psychometric chart can be approximated by a straight line [28]. It is also assumed that the fin is made of material with a constant thermal conductivity.

The governing energy equation for one-dimensional temperature distribution in dry fins, fully wet fins, and partially wet fins can be written, separately under steady-state conditions, as follows:

Dry surface

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y^p(r_i+x)^s\frac{\mathrm{d}T}{\mathrm{d}x}\right]=\frac{ph}{k}(r_i+x)^s y^{p-1}(T-T_a) \tag{1a}$$

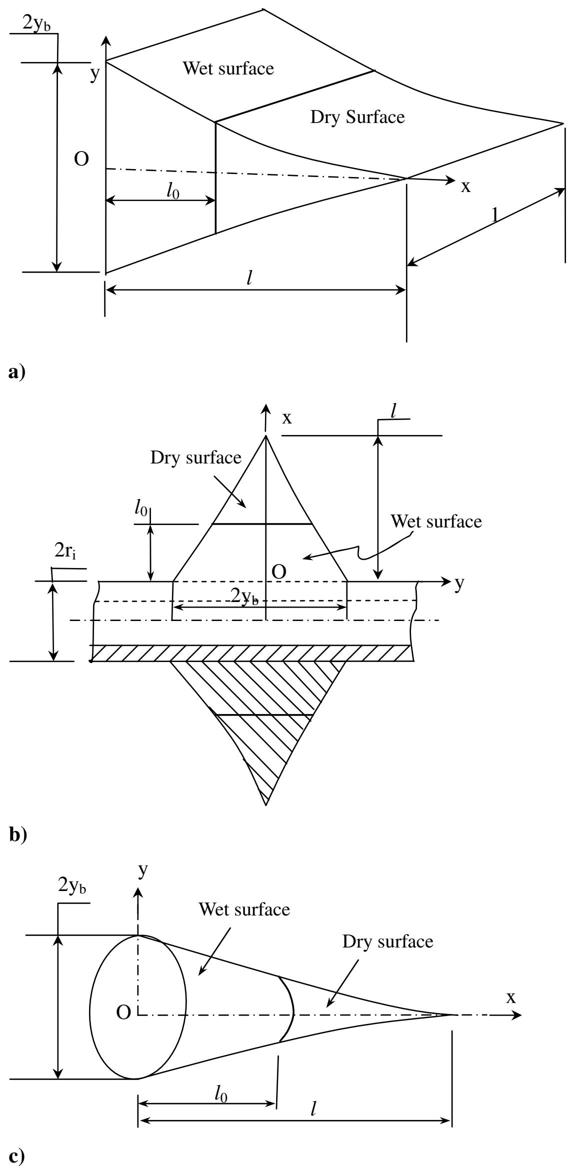

Fig. 1 Typical configuration of partially wet fins: a) longitudinal fin, b) annular fin, and c) spine.

Fully wet surface

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y^p(r_i+x)^s\frac{\mathrm{d}T}{\mathrm{d}x}\right]=\frac{ph}{k}(r_i+x)^s y^{p-1}\left[T-T_a+\frac{h_m(\omega-\omega_a)h_{fg}}{h}\right] \tag{1b}$$

Partially wet surface

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x}\left\{y^p(r_i+x)^s\frac{\mathrm{d}T}{\mathrm{d}x}\right\} \\ \frac{\mathrm{d}}{\mathrm{d}x}\left\{y^p(r_i+x)^s\frac{\mathrm{d}T}{\mathrm{d}x}\right\} \end{bmatrix} = \begin{bmatrix} \frac{ph}{k}(r_i+x)^s y^{p-1}(T-T_a) \\ \frac{ph}{k}(r_i+x)^s y^{p-1}\left\{T-T_a+\frac{h_m(\omega-\omega_a)h_{fg}}{h}\right\} \end{bmatrix} \tag{1c}$$

for dry domain $T \geq T_d$

for wet domain $T \leq T_d$

where

$$
\begin{aligned}
p &= 1, \quad s = 1 \quad \text{is valid for longitudinal fin} \\
p &= 2, \quad s = 0 \quad \text{is valid for spine} \\
p &= 1, \quad s = 1 \quad \text{is valid for annular fin}
\end{aligned} \tag{2}
$$

where h_m is the average mass transfer coefficient based on the humidity ratio difference, ω is the humidity ratio of saturated air at fin surface temperature T, ω_a is the humidity ratio of the atmospheric air, and h_{fg} is the latent heat of condensation.

For the mathematical simplicity, the following dimensionless variables and parameters are introduced:

$$
\begin{gathered}
X = hx/k, \qquad Y = hy/k, \qquad R_i = hr_i/k, \qquad L = hl/k \\
\theta = (T_a - T)/(T_a - T_b) \quad \text{and} \quad Le = (h/h_m C_p)^{3/2}
\end{gathered} \tag{3}
$$

where Le is the Lewis number. The relationship between heat and mass transfer coefficients is obtained from the Chilton–Colburn analogy [36]. The relationship between the saturated water film temperature T and the corresponding saturated humidity ratio ω can be approximated by a linear function [27], as given by

$$
\omega = a + bT \tag{4}
$$

where a and b are constants evaluated, respectively, from the saturated conditions of air at the fin base and fin tip for the fully wet surface, and from the saturated conditions of air at the fin base and the corresponding dew point of the surrounding air for the partially wet fin.

Equation (1) is normalized by using Eqs. (3) and (4) which yields

Dry surface

$$
\frac{\mathrm{d}}{\mathrm{d}X}\left[Y^p (R_i + X)^s \frac{\mathrm{d}\theta}{\mathrm{d}X}\right] = p(R_i + X)^s Y^{p-1}\theta \tag{5a}
$$

Fully wet surface

$$
\frac{\mathrm{d}}{\mathrm{d}X}\left[Y^p (R_i + X)^s \frac{\mathrm{d}\phi}{\mathrm{d}X}\right] = p(R_i + X)^s Y^{p-1}(1 + b\xi)\phi \tag{5b}
$$

Partially wet surface

$$
\begin{bmatrix}
\frac{\mathrm{d}}{\mathrm{d}X}\left\{Y^p (R_i + X)^s \frac{\mathrm{d}\theta}{\mathrm{d}X}\right\} \\
\frac{\mathrm{d}}{\mathrm{d}X}\left\{Y^p (R_i + X)^s \frac{\mathrm{d}\phi}{\mathrm{d}X}\right\}
\end{bmatrix}
=
\begin{bmatrix}
p(R_i + X)^s Y^{p-1}\theta \\
p(R_i + X)^s Y^{p-1}(1 + b\xi)\phi
\end{bmatrix} \tag{5c}
$$

for dry domain $\theta \geq \theta_d$
for wet domain $\theta \leq \theta_d$

where

$$
\begin{gathered}
\phi = \theta + \theta_p \\
\theta_p = (\omega_a - a - bT_a)/[(T_a - T_b)(1 + b\xi)] \quad \text{and} \\
\xi = h_{fg}/C_p Le^{2/3}
\end{gathered} \tag{6}
$$

It is assumed that the heat transfer through the tip is negligibly small compared with that leaving through the lateral surfaces for all the fin geometries, and fin base temperature is taken as a constant value. In the case of a partially wet surface, continuity of temperature and heat conduction satisfies at the section where dry and wet parts coexist. Thus, Eq. (5) is subjected to the following boundary conditions:

Dry surface

$$
\text{at } X = 0, \qquad \theta = 1 \tag{7a}
$$

$$
\text{at } X = L, \qquad Y^p \mathrm{d}\theta/\mathrm{d}X = 0 \tag{7b}
$$

Fully wet surface

$$
\text{at } X = 0, \qquad \phi = 1 + \theta_p = \phi_0 \tag{7c}
$$

$$
\text{at } X = L, \qquad Y^p \mathrm{d}\phi/\mathrm{d}X = 0 \tag{7d}
$$

Partially wet surface

$$
\text{at } X = 0, \qquad \phi = \phi_0 \tag{7e}
$$

$$
\text{at } X = L_0, \qquad \begin{cases} \theta = \theta_d \\ \mathrm{d}\theta/\mathrm{d}X = \mathrm{d}\phi/\mathrm{d}X \end{cases} \tag{7f}
$$

$$
\text{at } X = L, \qquad Y^p \mathrm{d}\theta/\mathrm{d}X = 0 \tag{7g}
$$

To determine the heat transfer duty through dry, fully wet, and partially wet fins, Eqs. (5a–5c) are multiplied by respective variables θ and ϕ, and then, integrating, the following relationships are obtained with the help of the corresponding boundary conditions as

Dry surface

$$
\begin{aligned}
&-[Y^p (R_i + X)^s \,\mathrm{d}\theta/\mathrm{d}X]_{X=0} \\
&\quad = \int_{X=0}^{L} Y^{p-1}(R_i + X)^s [Y(\mathrm{d}\theta/\mathrm{d}X)^2 + p\theta^2]\,\mathrm{d}X
\end{aligned} \tag{8a}
$$

Fully wet surface

$$
\begin{aligned}
&-[Y^p (R_i + X)^s \phi \,\mathrm{d}\phi/\mathrm{d}X]_{X=0} \\
&\quad = \int_{X=0}^{L} Y^{p-1}(R_i + X)^s [Y(\mathrm{d}\phi/\mathrm{d}X)^2 + p(1 + b\xi)\phi^2]\,\mathrm{d}X
\end{aligned} \tag{8b}
$$

Partially wet surface

$$
\begin{aligned}
&-[Y^p (R_i + X)^s \phi \,\mathrm{d}\phi/\mathrm{d}X]_{X=0} = -[Y^p (R_i + X)^s \phi \,\mathrm{d}\phi/\mathrm{d}X]_{X=L_0} \\
&\quad + \int_{X=0}^{L_0} Y^{p-1}(R_i + X)^s [Y(\mathrm{d}\phi/\mathrm{d}X)^2 + p(1 + b\xi)\phi^2]\,\mathrm{d}X
\end{aligned} \tag{8c}
$$

and

$$
\begin{aligned}
&-[Y^p (R_i + X)^s \theta \,\mathrm{d}\theta/\mathrm{d}X]_{X=L_0} \\
&\quad = \int_{X=L_0}^{L} Y^{p-1}(R_i + X)^s [Y(\mathrm{d}\theta/\mathrm{d}X)^2 + p\theta^2]\,\mathrm{d}X
\end{aligned} \tag{8d}
$$

Combining Eqs. (8c) and (8d) yields the following expressions:

$$
\begin{aligned}
&-[Y^p (R_i + X)^s \phi \,\mathrm{d}\phi/\mathrm{d}X]_{X=0} \\
&\quad = (\phi_d/\theta_d) \int_{X=L_0}^{L} Y^{p-1}(R_i + X)^s [Y(\mathrm{d}\theta/\mathrm{d}X)^2 + p\theta^2]\,\mathrm{d}X \\
&\quad + \int_{X=0}^{L_0} Y^{p-1}(R_i + X)^s [Y(\mathrm{d}\phi/\mathrm{d}X)^2 + p(1 + b\xi)\phi^2]\,\mathrm{d}X
\end{aligned} \tag{8e}
$$

The heat transfer rate through the fins is calculated by applying the Fourier's law of heat conduction at the fin base, and it can be written by using Eqs. (8a), (8b), and (8e), as

Dry surface

$$
\begin{aligned}
Q &= \frac{qp(h/\pi)^{p+s-1}}{(2s + 2)k^{p+s}(T_a - T_b)} = -\left[Y^p (R_i + X)^s \frac{\mathrm{d}\theta}{\mathrm{d}X}\right]_{X=0} \\
&= \int_{X=0}^{L} Y^{p-1}(R_i + X)^s \left[Y\left(\frac{\mathrm{d}\theta}{\mathrm{d}X}\right)^2 + p\theta^2\right]\mathrm{d}X
\end{aligned} \tag{9a}
$$

Fully wet surface

$$Q = \frac{qp(h/\pi)^{p+s-1}}{(2s+2)k^{p+s}(T_a - T_b)} = -\left[Y^p(R_i + X)^s \frac{\mathrm{d}\phi}{\mathrm{d}X}\right]_{X=0}$$
$$= \frac{1}{\phi_0}\int_{X=0}^{L} Y^{p-1}(R_i + X)^s\left[Y\left(\frac{\mathrm{d}\phi}{\mathrm{d}X}\right)^2 + p(1+b\xi)\phi^2\right]\mathrm{d}X \tag{9b}$$

Partially wet surface

$$Q = \frac{qp(h/\pi)^{p+s-1}}{(2s+2)k^{p+s}(T_a - T_b)} = -\left[Y^p(R_i + X)^s \frac{\mathrm{d}\phi}{\mathrm{d}X}\right]_{X=0}$$
$$= \frac{\phi_d}{\phi_0\theta_d}\int_{X=L_0}^{L} Y^{p-1}(R_i + X)^s\left[Y\left(\frac{\mathrm{d}\theta}{\mathrm{d}X}\right)^2 + p\theta^2\right]\mathrm{d}X$$
$$+ \frac{1}{\phi_0}\int_{X=0}^{L_0} Y^{p-1}(R_i + X)^s\left[Y\left(\frac{\mathrm{d}\phi}{\mathrm{d}X}\right)^2 + p(1+b\xi)\phi^2\right]\mathrm{d}X \tag{9c}$$

Volume of the longitudinal, spine, and annular fins cans be obtained from the expression given as follows:

$$U = \frac{Vp(h/k)^{p+s+1}}{(2s+2)\pi^{p+s-1}} = \int_{X=0}^{L}(R_i + X)^s Y^p \,\mathrm{d}X \tag{10}$$

A. Variational Principle

The profile shape of a fin is determined from the variational principle after satisfying the maximization of heat transfer rate Q for a constant design condition. In the present study, either the fin volume or both the fin volume and length are considered as a constraint condition. A functional F may be constructed from Eqs. (9) and (10) using Lagrange multiplier λ and thus,

Dry Surface

$$F = Q - \lambda U = \int_{X=0}^{L} Y^{p-1}(R_i + X)^s\left[Y\left(\frac{\mathrm{d}\theta}{\mathrm{d}X}\right)^2 + p\theta^2 - \lambda Y\right]\mathrm{d}X \tag{11a}$$

Fully wet surface

$$F = Q - \lambda U = \frac{1}{\phi_0}\int_{X=0}^{L} Y^{p-1}(R_i + X)^s\left[Y\left(\frac{\mathrm{d}\phi}{\mathrm{d}X}\right)^2 + p(1+b\xi)\phi^2 - \lambda\phi_0 Y\right]\mathrm{d}X \tag{11b}$$

Partially wet surface

$$F = Q - \lambda U = \frac{1}{\phi_0}\int_{X=0}^{L_0} Y^{p-1}(R_i + X)^s\left[Y\left(\frac{\mathrm{d}\phi}{\mathrm{d}X}\right)^2 + p(1+b\xi)\phi^2 - \lambda\phi_0 Y\right]\mathrm{d}X$$
$$+ \frac{\phi_d}{\phi_0\theta_d}\int_{X=L_0}^{L} Y^{p-1}(R_i + X)^s\left[Y\left(\frac{\mathrm{d}\theta}{\mathrm{d}X}\right)^2 + p\theta^2 - \frac{\lambda\phi_0\theta_d Y}{\phi_d}\right]\mathrm{d}X \tag{11c}$$

From the preceding equations, the relation between the variation of F and that of Y is obtained and, for maximizing value of F, δF is set to zero for any admissible variation of δY. Thus,

Dry Surface

$$\delta F = \int_{X=0}^{L} pY^{p-2}(R_i + X)^s\left[Y\left(\frac{\mathrm{d}\theta}{\mathrm{d}X}\right)^2 + (p-1)\theta^2 - \lambda Y\right]\delta Y\,\mathrm{d}X = 0 \tag{12a}$$

Fully wet surface

$$\delta F = \frac{1}{\phi_0}\int_{X=0}^{L} pY^{p-2}(R_i + X)^s\left[Y\left(\frac{\mathrm{d}\phi}{\mathrm{d}X}\right)^2 + (p-1)(1+b\xi)\phi^2 - \lambda\phi_0 Y\right]\delta Y\,\mathrm{d}X = 0 \tag{12b}$$

Partially wet surface

$$\delta F = \frac{1}{\phi_0}\int_{X=0}^{L_0} pY^{p-2}(R_i + X)^s\left[Y\left(\frac{\mathrm{d}\phi}{\mathrm{d}X}\right)^2 + (p-1)(1+b\xi)\phi^2 - \lambda\phi_0 Y\right]\delta Y\,\mathrm{d}X + \frac{\phi_d}{\phi_0\theta_d}\int_{X=L_0}^{L} pY^{p-2}(R_i + X)^s\left[Y\left(\frac{\mathrm{d}\theta}{\mathrm{d}X}\right)^2 + (p-1)\theta^2 - \frac{\lambda\phi_0\theta_d Y}{\phi_d}\right]\delta Y\,\mathrm{d}X = 0 \tag{12c}$$

Therefore, from Eq. (12), the following optimality criteria are obtained:

Dry fin

$$Y(\mathrm{d}\theta/\mathrm{d}X)^2 + (p-1)\theta^2 - \lambda Y = 0 \tag{13a}$$

Fully wet fin

$$Y(\mathrm{d}\phi/\mathrm{d}X)^2 + (p-1)(1+b\xi)\phi^2 - \lambda\phi_0 Y = 0 \tag{13b}$$

Partially wet fin

$$\begin{bmatrix} Y(\mathrm{d}\theta/\mathrm{d}X)^2 + (p-1)\theta^2 - \lambda\phi_0\theta_d\, Y/\phi_d \\ Y(\mathrm{d}\phi/\mathrm{d}X)^2 + (p-1)(1+b\xi)\phi^2 - \lambda\phi_0 Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix} \theta > \theta_d \\ \theta \le \theta_d \end{matrix} \tag{13c}$$

From Eq. (13), it is clear that the temperature gradient in both the longitudinal and radial fins ($p = 1$) at the optimum condition is a constant, and, for radial fins, it is independent of the constant factor "s" irrespective of their surface conditions. It may be noted that the temperature gradient in spines cannot be predicted by using only the optimality criteria expressed in Eq. (13). Therefore, in the case of a spine, the following mathematical steps are provided for evaluating the temperature gradient.

As the optimality criterion is independent of the factor s, the governing energy Eq. (5) for the spine can be written with the consideration of $s = 0$ as

Dry surface

$$\mathrm{d}\,(Y^p\,\mathrm{d}\theta/\mathrm{d}X)/\mathrm{d}X = pY^{p-1}\theta \tag{14a}$$

Fully wet surface

$$\mathrm{d}\,(Y^p\,\mathrm{d}\phi/\mathrm{d}X)/\mathrm{d}X = pY^{p-1}(1+b\xi)\phi \tag{14b}$$

Partially wet surface

$$\begin{bmatrix} \mathrm{d}(Y^p\,\mathrm{d}\theta/\mathrm{d}X)/\mathrm{d}X \\ \mathrm{d}(Y^p\,\mathrm{d}\phi/\mathrm{d}X)/\mathrm{d}X \end{bmatrix} = \begin{bmatrix} pY^{p-1}\theta \\ pY^{p-1}(1+b\xi)\phi \end{bmatrix} \tag{14c}$$
$$\text{for dry domain } \theta \ge \theta_d$$
$$\text{for wet domain } \theta \le \theta_d$$

Multiplying both sides of Eq. (14) by respective $\mathrm{d}\theta/\mathrm{d}X$ and $\mathrm{d}\phi/\mathrm{d}X$ yields the following expressions after some manipulations:

Dry surface

$$\mathrm{d}\,[Y^p(\mathrm{d}\theta/\mathrm{d}X)^2]/\mathrm{d}X + (\mathrm{d}\theta/\mathrm{d}X)^2\,\mathrm{d}(Y^p)/\mathrm{d}X = pY^{p-1}\,\mathrm{d}(\theta^2)/\mathrm{d}X \tag{15a}$$

Fully wet surface

$$\mathrm{d}[Y^p(\mathrm{d}\phi/\mathrm{d}X)^2]/\mathrm{d}X + (\mathrm{d}\phi/\mathrm{d}X)^2\,\mathrm{d}(Y^p)/\mathrm{d}X = p(1+b\xi)Y^{p-1}\,\mathrm{d}(\phi^2)/\mathrm{d}X \tag{15b}$$

Partially wet surface

$$\begin{bmatrix} \mathrm{d}\{Y^p(\mathrm{d}\theta/\mathrm{d}X)^2\}/\mathrm{d}X + (\mathrm{d}\theta/\mathrm{d}X)^2\,\mathrm{d}(Y^p)/\mathrm{d}X \\ \mathrm{d}\{Y^p(\mathrm{d}\phi/\mathrm{d}X)^2\}/\mathrm{d}X + (\mathrm{d}\phi/\mathrm{d}X)^2\,\mathrm{d}(Y^p)/\mathrm{d}X \end{bmatrix} = \begin{bmatrix} pY^{p-1}\,\mathrm{d}(\theta^2)/\mathrm{d}X \\ p(1+b\xi)Y^{p-1}\,\mathrm{d}(\phi^2)/\mathrm{d}X \end{bmatrix} \quad \begin{matrix}\theta \geq \theta_d \\ \theta \leq \theta_d\end{matrix} \tag{15c}$$

Using Eq. (13) to eliminate respective θ^2 and ϕ^2 from Eq. (15) and then integrating, the following equations are obtained:

Dry surface

$$(2p-1)Y^p(\mathrm{d}\theta/\mathrm{d}X)^2 - \lambda Y^p = C_1 \quad \text{for } p \neq 1 \tag{16a}$$

Fully wet surface

$$(2p-1)Y^p(\mathrm{d}\phi/\mathrm{d}X)^2 - \lambda\phi_0 Y^p = C_2 \quad \text{for } p \neq 1 \tag{16b}$$

Partially wet surface

$$\begin{bmatrix} (2p-1)Y^p(\mathrm{d}\theta/\mathrm{d}X)^2 - \lambda\phi_0\theta_d Y^p/\phi_d \\ (2p-1)Y^p(\mathrm{d}\phi/\mathrm{d}X)^2 - \lambda\phi_0 Y^p \end{bmatrix} = \begin{bmatrix} C_3 \\ C_4 \end{bmatrix} \quad \text{for } p \neq 1 \tag{16c}$$

where Cs are the integration constants determined from the boundary conditions.

B. Optimum Fin Shape for the Volume Constraint

As there is no constraint on the length L, it can be taken as a variable. The variation of F with a function of L [from Eq. (11)] gives the following results:

Dry surface

$$\delta F = Y^{p-1}(R_i + X)^s[Y(\mathrm{d}\theta/\mathrm{d}X)^2 + p\theta^2 - \lambda Y]\delta X|_{X=0}^{L} = 0 \tag{17a}$$

Fully wet surface

$$\delta F = \frac{Y^{p-1}(R_i+X)^s}{\phi_0}\left[Y\left(\frac{\mathrm{d}\phi}{\mathrm{d}X}\right)^2 + p(1+b\xi)\phi^2 - \lambda\phi_0 Y\right]\delta X|_{X=0}^{L} = 0 \tag{17b}$$

Partially wet surface

$$\begin{bmatrix}\delta F \\ \delta F\end{bmatrix} = \begin{bmatrix} \frac{Y^{p-1}(R_i+X)^s}{\phi_0}\left\{Y\left(\frac{\mathrm{d}\phi}{\mathrm{d}X}\right)^2 + p(1+b\xi)\phi^2 - \lambda\phi_0 Y\right\}\delta X|_{X=0}^{X=L_0} \\ \frac{Y^{p-1}(R_i+X)^s\phi_d}{\theta_d\phi_0}\left\{Y\left(\frac{\mathrm{d}\theta}{\mathrm{d}X}\right)^2 + p\theta^2 - \lambda Y\phi_0\theta_d/\phi_d\right\}\delta X|_{X=L_0}^{L} \end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \quad \begin{matrix}\theta \leq \theta_d \\ \theta \geq \theta_d\end{matrix} \tag{17c}$$

At $X = 0$, the preceding term must vanish as $\delta X = 0$. At $X = L_0$ and $X = L$, δX is not zero; therefore, at the location where dry and wet surfaces are present, and at the tip, the following optimality conditions are obtained:

Dry surface

$$Y(\mathrm{d}\theta/\mathrm{d}X)^2 + p\theta^2 - \lambda Y = 0 \tag{18a}$$

Fully wet surface

$$Y(\mathrm{d}\phi/\mathrm{d}X)^2 + p(1+b\xi)\phi^2 - \lambda\phi_0 Y = 0 \tag{18b}$$

Partially wet surface

$$\begin{bmatrix} Y(\mathrm{d}\theta/\mathrm{d}X)^2 + p\theta^2 - \lambda\theta_d\phi_0 Y/\phi_d \\ Y(\mathrm{d}\phi/\mathrm{d}X)^2 + p(1+b\xi)\phi^2 - \lambda\phi_0 Y \end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \quad \text{at } X = L_0 \tag{18c}$$

and

$$Y(\mathrm{d}\theta/\mathrm{d}X)^2 + p\theta^2 - \lambda\theta_d\phi_0 Y/\phi_d = 0 \quad \text{at } X = L \tag{18d}$$

Combining Eqs. (5), (7), (13), and (18), respectively, yields the tip temperature as follows:

$$\begin{bmatrix}\theta\\\phi\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \quad \begin{matrix}\text{for dry and partially wet surfaces} \\ \text{for fully wet surface}\end{matrix} \tag{19}$$

The tip thickness of a fin is now determined from the tip temperature and the optimality conditions [Eqs. (18) and (19)]. It can be seen that, for the three types of examined fins, tip thickness becomes zero irrespective of surface conditions. In the case of tip temperature, it depends upon the surface conditions. For the dry and partially wet surfaces, temperature at the tip is equal to the ambient temperature, whereas, for the fully wet surface, it may not be the same as that for fully dry and partially wet, which may be slightly less than the ambient value ($\theta_t = -\theta_p$). Moreover, this temperature is obvious as a function of psychometric properties of the surrounding fluid.

Using Eqs. (7), (13), (16), and (19), the temperature distribution and fin profile for the three types of fins can be expressed in a generalized form as

Dry surface

$$\theta = 1 - X/L \tag{20a}$$

and

$$Y = \frac{1}{2\cdot 3^s(R_i+X)^s}[s(L^3 - 3LX^2 + 2X^3) + 3^s R_i^s(L-X)^2] \tag{20b}$$

Fully wet surface

$$\theta = 1 - (1+\theta_p)X/L \tag{20c}$$

and

$$Y = \frac{(1+b\xi)}{2\cdot 3^s(R_i+X)^s}[s(L^3 - 3LX^2 + 2X^3) + 3^s R_i^s(L-X)^2] \tag{20d}$$

Partially wet surface

$$\theta = \begin{cases} 1 - (1-\theta_d)X/L_0 & 0 \leq X \leq L_0 \\ \theta_d(L-X)/(L-L_0) & L_0 \leq X \leq L \end{cases} \tag{20e}$$

and

$$\begin{aligned} Y = \frac{1}{2\cdot 3^s(R_i+X)^s}\Bigg[&6^s L^s R_i^s(L-L_0)^{2-s} \\ &+ 3s(L-R_i)(L^2-L_0^2) - 2s(L^3-L_0^3) \\ &+ \frac{(1+b\xi)}{(1-\theta_d)}\{2\cdot 3^s R_i^s L_0\phi_0(L_0-X) \\ &+ 3^s[s\phi_0 L_0 - (1-\theta_d)R_i^s](L_0^2 - X^2) \\ &- 2s(1-\theta_d)(L_0^3 - X^3)\}\Bigg] \quad \text{for } 0 \leq X \leq L_0 \end{aligned} \tag{20f}$$

$$Y = \frac{1}{2\cdot 3^s (R_i + X)^s}[6^s L^s R_i^s (L-X)^{2-s} + 3s(L-R_i)(L^2-X^2) - 2s(L^3 - X^3)] \quad \text{for } X_0 \le X \le L \tag{20g}$$

The length of the wet region L_0 is determined from an energy balance at the section where dry and wet parts coexist as

$$L_0 = L(1-\theta_d) \tag{20h}$$

Here, L is a variable. Therefore, L can be obtained from Eqs. (10) and (20) for the dry, fully wet, and partially wet surface conditions of a fin. Thus, the optimum length and the maximum heat transfer rate through the fin can be expressed in the design variables as follows:

Dry surface

$$\begin{bmatrix} L_{\text{opt}} \\ Q_{\max} \end{bmatrix} = \begin{bmatrix} \{2^p(2p+1)U\}^{1/(2p+1)} \\ \{p(2p+1)^{2p-1}U^{2p-1}/4^p\}^{1/(2p+1)} \end{bmatrix} \tag{21a}$$

for longitudinal and spine

$$\begin{bmatrix} L_{\text{opt}} \\ Q_{\max} \end{bmatrix} = \begin{bmatrix} L_{\text{opt}}^4 + 2R_i L_{\text{opt}}^3 - 12U = 0 \\ L_{\text{opt}}(3R_i - L_{\text{opt}})/6 \end{bmatrix} \quad \text{for annular fins} \tag{21b}$$

Fully wet surface

$$\begin{bmatrix} L_{\text{opt}} \\ Q_{\max} \end{bmatrix} = \begin{bmatrix} \{2^p(2p+1)U/(1+b\xi)^p\}^{1/(2p+1)} \\ \phi_0\{p(2p+1)^{2p-1}U^{2p-1}(1+b\xi)^{2p}/4^p\}^{1/(2p+1)} \end{bmatrix}$$

for longitudinal and spine (21c)

$$\begin{bmatrix} L_{\text{opt}} \\ Q_{\max} \end{bmatrix} = \begin{bmatrix} L_{\text{opt}}^4 + 2R_i L_{\text{opt}}^3 - 12U/(1+b\xi) = 0 \\ \phi_0 L_{\text{opt}}(3R_i - L_{\text{opt}})(1+b\xi)/6 \end{bmatrix} \tag{21d}$$

for annular fins

To determine the optimum length of an annular fin for the dry and fully wet conditions, Eqs. (21b) and (21d) have been solved by using the Newton–Raphson iterative method [37] for a given R_i and U.

Partially wet

The length and heat transfer rate for longitudinal and pin fins have been determined for a volume constraint as

$$L_{\text{opt}} = \frac{[(2^{2(p+1)} + 6p - 16)U]^{1/(2p+1)}}{[(2p-1)\theta_d^{2p}(2p+1-2\theta_d) + Z_1\{(p-1)Z_2 + 10\theta_d^2(p-1)+1\}(3\phi_0 + 2\theta_d - 2)]^{1/(2p+1)}} \tag{21e}$$

$$Q_{\max} = \frac{[\theta_d^2 + (1+b\xi)(2\phi_0+\theta_d-1)(1-\theta_d)]^2[(2^{2(p+1)}+6p-16)U]^{(2p-1)/(2p+1)}}{[(2p-1)\theta_d^{2p}(2p+1-2\theta_d) + Z_1\{(p-1)Z_2 + 10\theta_d^2(p-1)+1\}(3\phi_0+2\theta_d-2)]^{(2p-1)/(2p+1)}} \tag{21f}$$

where

$$Z_1 = (1+b\xi)(1-\theta_d)^2 \tag{21g}$$

and

$$Z_2 = (1-\theta_d)(1+b\xi)\{5(2\phi_0+\theta_d-1)(5-6\phi_0-2\theta_d) + 5\phi_0(4\phi_0+3\theta_d-3) + 3(1-\theta_d)^2\} \tag{21h}$$

In case of annular fins, the design variables cannot be predicted analytically. An algebraic equation for the optimality condition for annular fins is obtained by combining fin volume and fin profile as

$$[2\theta_d^2(1-\theta_d) + (1+b\xi)\{4\phi_0 - 4\theta_d\phi_0 - 3(1-\theta_d)^2\}(1-\theta_d)^2 + \theta_d^3(2-\theta_d)]L^4 + [6\theta_d^2(1-\theta_d) + 2(1+b\xi)(3\phi_0 + 2\theta_d - 2)(1-\theta_d)^2 + 2\theta_d^3]R_i L^3 - 12U = 0 \tag{21i}$$

The preceding equation can be solved by the Newton–Raphson iterative method [37] to get the optimum length of the fin L_{opt}. After estimating the L_{opt} value, one can calculate the maximum heat transfer rate, which can be determined from the expression give as follows:

$$Q_{\max} = \frac{L_{\text{opt}}}{6}[6R_i\theta_d + 3(L_{\text{opt}} - R_i)\{1-(1-\theta_d)^2\} - 2L_{\text{opt}}\{1-(1-\theta_d)^3\} + (1-\theta_d)(1+b\xi)L_{\text{opt}}^2\{6R_i\phi_0 + 3(\phi_0 L_{\text{opt}} - R_i)(1-\theta_d) - 2L_{\text{opt}}(1-\theta_d)^2\}] \tag{21j}$$

C. Optimum Fin Shape for Both Length and Volume Constraints

In a fin design problem, the length of the fin is specified due to the restriction of space as well as ease of fabrication. In that situation of the optimization, both the length (fixed L) and volume may be considered as a constraint. To establish the analysis of this application, the following mathematical steps are provided.

From Eqs. (7b), (7d), (7g), and (13), it can be noted that the fin thickness at the tip for the optimum longitudinal and radial fins becomes zero. However, in the case of spines, the fin thickness at the tip may not be equal to zero. The temperature distribution and fin profile for the longitudinal and annular fins can be determined by using Eqs. (7), (10), and (13), as

Dry surface

$$\theta = 1 - \alpha X \tag{22a}$$

$$Y = \frac{1}{2\cdot 3^s\alpha(R_i+X)^s}[2\cdot 3^s R_i^s(L-X) + 3^s(s-\alpha R_i^s)(L^2-X^2) - 2s\alpha(L^3-X^3)] \tag{22b}$$

where

$$\alpha = \frac{2^s(3R_i^s + 2sL)L^2}{6U(1+s) + [2(1-s) + 4sR_i + 3sL]L^3} \tag{22c}$$

Fully wet surface

$$\theta = 1 - \alpha X \tag{22d}$$

$$Y = \frac{(1+b\xi)}{2\cdot 3^s\alpha(R_i+X)^s}[2\cdot 3^s R_i^s\phi_0(L-X) + 3^s(s\phi_0 - \alpha R_i^s)(L^2-X^2) - 2s\alpha(L^3-X^3)] \tag{22e}$$

where

$$\alpha = \frac{2^s(3R_i^s + 2sL)L^2(1+b\xi)\phi_0}{6U(1+s) + L^3(1+b\xi)[2(1-s) + 4sR_i + 3sL]} \tag{22f}$$

Partially wet fin

$$\begin{bmatrix}\theta\\ \theta\end{bmatrix}=\begin{bmatrix}1-\alpha X\\ \theta_d-\alpha(X-L_0)\end{bmatrix}\qquad \begin{matrix}0\le X\le L_0\\ L_0\le X\le L\end{matrix} \tag{22g}$$

$$\begin{aligned}Y=&\frac{1}{2\cdot 3^s\alpha(R_i+X)^s}[2\cdot 3^sR_i^s(\theta_d+\alpha L_0)(L-L_0)\\&+3^s(s\theta_d-\alpha R_i^s+s\alpha L_0)(L^2-L_0^2)-2s\alpha(L^3-L_0^3)\\&+(1+b\xi)\{2\cdot 3^s\phi_0R_i^s(L_0-X)+3^s(s\phi_0-\alpha R_i^s)(L_0^2-X^2)\\&-2s\alpha(L_0^3-X^3)\}],\qquad 0\le X\le L_0\end{aligned} \tag{22h}$$

$$\begin{aligned}Y=&\frac{1}{2\cdot 3^s\alpha(R_i+X)^s}[2\cdot 3^s(R_i^s\theta_d+\alpha R_i^sL_0)(L-X)\\&+3^s(s\alpha L_0-\alpha R_i^s+s\theta_d)(L^2-X^2)\\&-2s\alpha(L^3-X^3)]\qquad L_0\le X\le L\end{aligned} \tag{22i}$$

where

$$\alpha=\frac{3(p+s)R_i^s\theta_d(L^2-L_0^2)+\phi_0(3-s)(1+b\xi)(3R_i+2L_0)^sL_0^2+4s\theta_d(L^3-L_0^3)}{6(p+s)U+L_0^3(2-s)(1+b\xi)(4R_i+3L_0)^s-2sR_iL_0(3L^2-L_0^2)+L_k} \tag{22j}$$

$$L_0=(1-\theta_d)/\alpha \tag{22k}$$

and

$$\begin{aligned}L_k=&L^3(2-s)(4R_i+3L)^s-L_0[(3+s)L^{2+s}-L_0^{2+s}]\\&+4(1-s)(L^3-L_0^3)\end{aligned} \tag{22l}$$

Combining Eqs. (22j) and (22k), the following transcendental equation is obtained:

$$\begin{aligned}&[(1+b\xi)\{2+s-(2+s)\theta_d-(3+s)\phi_0\}+3(2-s)\theta_d\\&+(4s-3)]L_0^{3+s}+2R_is[(1-\theta_d)(3+2b\xi)-3\phi_0(1+b\xi)\\&+3\theta_d]L_0^3-(3-s)L^2L_0+(1-\theta_d)\{6(s+1)U+4sR_iL^3\\&+3(2-s)L^{3+s}\}=0\end{aligned} \tag{22m}$$

To determine the wet length L_0, Eq. (22m) is solved by the Newton–Raphson iterative method [37].

In the case of a spine, the temperature distribution and fin profile can be found from Eqs. (5), (6), and (16) for the different surface conditions, as follows:

Dry surface

$$\theta=\sqrt{Y_b(Y_t^2Y^{-1}+2Y)/(Y_t^2+2Y_b^2)},\qquad Y_t\le Y\le Y_b \tag{23a}$$

and

$$\int_{Y_t}^{Y}\frac{(2Y^2-Y_t^2)\,\mathrm{d}Y}{\sqrt{(Y^2-Y_t^2)(Y_t^2+2Y^2)Y}}=2(L-X) \tag{23b}$$

Fully wet surface

$$\phi=\phi_0\sqrt{Y_b(Y_t^2Y^{-1}+2Y)/(Y_t^2+2Y_b^2)},\qquad Y_t\le Y\le Y_b \tag{23c}$$

and

$$\int_{Y_t}^{Y}\frac{(2Y^2-Y_t^2)\,\mathrm{d}Y}{\sqrt{(Y^2-Y_t^2)(Y_t^2+2Y^2)Y}}=2\sqrt{(1+b\xi)}(L-X) \tag{23d}$$

Partially wet surface

$$\phi=\phi_0\sqrt{\frac{Y_b}{Y}\left[\frac{2\phi_dY^2+(\phi_d-\theta_d)Y_0^2+\theta_dY_t^2}{2\phi_dY_b^2+(\phi_d-\theta_d)Y_0^2+\theta_dY_t^2}\right]},\qquad Y_0\le Y\le Y_b \tag{23e}$$

$$\theta=\theta_d\sqrt{Y_0(Y_t^2Y^{-1}+2Y)/(Y_t^2+2Y_0^2)},\qquad Y_0\le Y\le Y_t \tag{23f}$$

$$\int_{Y_0}^{Y}\frac{(2Y^2-Y_0^2)\,\mathrm{d}Y}{\sqrt{(Y^2-Y_0^2)(Y_0^2+2Y^2)Y}}=2\sqrt{(1+b\xi)}(L_0-X) \tag{23g}$$

and

$$\int_{Y_t}^{Y}\frac{(2Y^2-Y_t^2)\,\mathrm{d}Y}{\sqrt{(Y^2-Y_t^2)(Y_t^2+2Y^2)Y}}=2(L-X) \tag{23h}$$

From Eq. (23), it is clear that the temperature distribution and the fin profile for spines may be obtained if Y_b and Y_t are known, irrespective of their surface conditions, however, for the partially wet condition, one additional dimension Y_0 is essential. Thus, to calculate these unknowns, two constraint equations are provided for dry and fully wet surface conditions. However, in the case of a partially wet surface, the additional conditions (continuity of both temperature and heat conduction) at the section at which dry and wet parts coexist are required to determine Y_0. Therefore,

Dry surface

$$U=\int_{Y=Y_t}^{Y_b}\frac{(2Y^2-Y_t^2)Y^2\,\mathrm{d}Y}{2\sqrt{(Y^2-Y_t^2)(Y_t^2+2Y^2)Y}} \tag{24a}$$

and

$$\int_{Y=Y_t}^{Y_b}\frac{(2Y^2-Y_t^2)\,\mathrm{d}Y}{\sqrt{(Y^2-Y_t^2)(Y_t^2+2Y^2)Y}}=2L \tag{24b}$$

Fully wet surface

$$U=\int_{Y=Y_t}^{Y_b}\frac{(2Y^2-Y_t^2)Y^2\,\mathrm{d}Y}{2\sqrt{(1+b\xi)}\sqrt{(Y^2-Y_t^2)(Y_t^2+2Y^2)Y}} \tag{24c}$$

and

$$\int_{Y=Y_t}^{Y_b}\frac{(2Y^2-Y_t^2)\,\mathrm{d}Y}{\sqrt{(Y^2-Y_t^2)(Y_t^2+2Y^2)Y}}=2\sqrt{(1+b\xi)}L \tag{24d}$$

Partially wet surface

$$\begin{aligned}U=&\int_{Y=Y_0}^{Y_b}\frac{(2Y^2-Y_0^2)Y^2\,\mathrm{d}Y}{2\sqrt{(1+b\xi)}\sqrt{(Y^2-Y_0^2)(Y_0^2+2Y^2)Y}}\\&+\int_{Y=Y_t}^{Y_0}\frac{(2Y^2-Y_t^2)Y^2\,\mathrm{d}Y}{2\sqrt{(Y^2-Y_t^2)(Y_t^2+2Y^2)Y}}\end{aligned} \tag{24e}$$

and

$$
\int_{Y=Y_0}^{Y_b} \frac{(2Y^2 - Y_0^2)\,\mathrm{d}Y}{\sqrt{(Y^2 - Y_0^2)(Y_0^2 + 2Y^2)Y}} + \int_{Y=Y_t}^{Y_0} \frac{(2Y^2 - Y_t^2)\,\mathrm{d}Y}{\sqrt{(Y^2 - Y_t^2)(Y_t^2 + 2Y^2)Y}} = 2\sqrt{(1 + b\xi)}L_0 + 2(L - L_0) \tag{24f}
$$

To determine Y_b, Y_0, and Y_t, the preceding set of equations [Eq. (24)] can be solved simultaneously.

III. Results and Discussion

Based on the preceding formulations, results have been taken for a wide range of thermopsychometric parameters. As the humid air is a mixture of dry air and water vapor, three thermodynamic properties of air are essential to identify a state point, which is used for calculating the results to show the effect of dehumidification of air on the optimizing shape of a fin. In general, air properties such as pressure, temperature, and relative humidity of air are used as a thermodynamic property. In the present study, the effect of these psychometric properties of air on the present optimization scheme has been investigated. From the design point of view, two constraints, namely, fin volume, and both fin volume and length, have been adopted to furnish the result.

As one of the main objectives is to determine the profile shape and temperature distribution in an optimum fin under wet conditions, Fig. 2 is depicted as a function of fin length for a constraint fin volume. To make a comparison between dry and wet surface fins, the preceding results for the dry fin have also been evaluated and have been plotted in the same figure. From the mathematical formulation, it is noticeable that the temperature distribution in the fin at an optimum condition under dry, partially wet, and fully wet surfaces varies linearly, which is displayed in Fig. 2a. For the same temperature specified at the fin base for all the surface conditions considered here, temperature on the fin surface for wet fins differs from that of the dry surface fin, and this discrepancy increases gradually from the fin base to fin tip. The difference in temperature occurs due to evolving of latent heat of condensation of moisture on the fin surface. This difference in temperature is found to be a maximum value when the relative humidity of air becomes 100% because, with this relative humidity, the amount of condensation of moisture is a maximum value and, as a result, latent heat released is also the highest. From the mathematical expression of temperature distribution, it can be highlighted that, for the dry and partially wet fin at an optimum condition, the tip temperature is equal to the ambient temperature. However, in the case of a fully wet fin, the tip temperature may be slightly less than the ambient value, however, it depends upon the psychrometric condition of the design. Figure 2b depicts the optimum fin profile for dry, partially wet, and fully wet conditions. For the dry fin, the optimum length and optimum base thickness are larger and smaller, respectively, than that for the wet fin for the same fin volume, and this difference is a maximum value for a fully wet fin of 100% relative humidity of air. Nevertheless, the effect of relative humidity on the optimum fin profile for a wet fin is shown marginally. The same nature of temperature distribution and the optimum fin profile for the annular and pin fins are noticed, which is shown in Figs. 3 and 4, respectively. In addition, it can be mentioned that, in the case of annular fins, the temperature distribution and fin profile are also a function of the tube outer radius parameter R_i.

The effect of fin base and ambient temperature of air on the design parameters of an optimum longitudinal fin under wet conditions for the constraint fin volume has been examined, and is graphically shown in Fig. 5. For this observation, the relative humidity of air is taken as 70% arbitrarily. At this relative humidity, a partially wet surface is obtained. From the figure, it is noted that, for a constant ambient temperature, an increase in base temperature increases the optimum fin length, however, with the increase in ambient temperature for a constant base temperature, the optimum fin length decreases. The length of the wet part L_0 not only depends on the relative humidity but also depends upon the other properties, such as ambient temperature and base temperature, and this result is presented in Fig. 5b. For a constant relative humidity, increasing ambient temperature increases the vapor pressure of moisture and thus increases the dew point temperature of the surrounding air, and, as a result, the wet part increases. The length of the wet part decreases with the increase in base temperature, which is shown in Fig. 5b. It

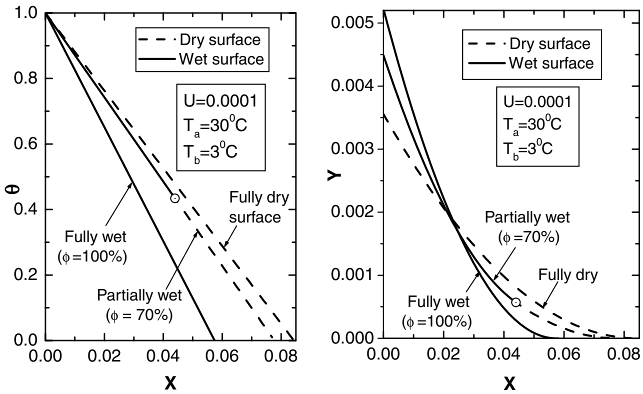

Fig. 2 Temperature distribution and fin profile of an optimum longitudinal fin as a function of fin length for the different surface conditions under a volume constraint: a) fin temperature, and b) fin profile.

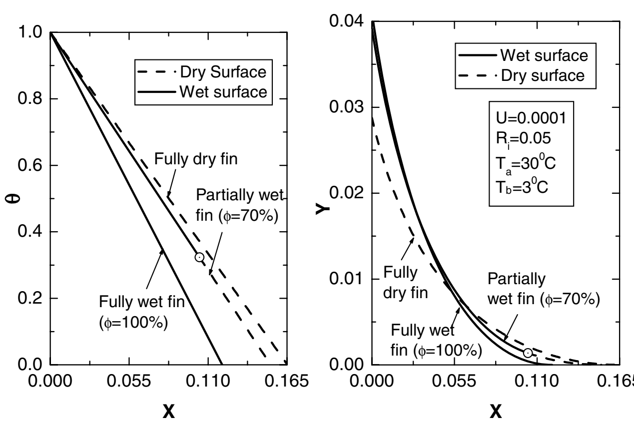

Fig. 3 Temperature distribution and fin profile of an optimum annular fin as a function of fin length for the different surface conditions under a volume constraint: a) fin temperature, and b) fin profile.

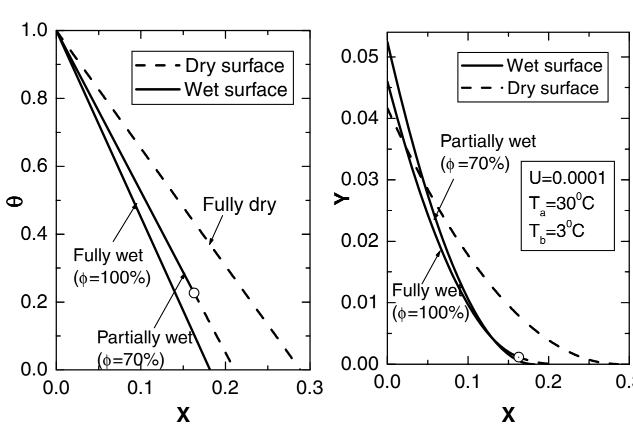

Fig. 4 Temperature distribution and fin profile of an optimum spine as a function of fin length for the different surface conditions under a volume constraint: a) fin temperature, and b) fin profile.

can be explained that, with the increase in base temperature, the tip temperature increases for a constant ambient temperature and relative humidity. This effect may cause an increase in the dry part in a partially wet fin. On the other hand, the effect of the ambient and base temperature on the base thickness and heat transfer rate is depicted in Figs. 5c and 5d, respectively. For a constant base temperature, both the base thickness and heat transfer rate increase with the ambient temperature, whereas, for a constant ambient temperature, these design parameters show a peculiarity in nature with the base temperature, and this nature, however, depends upon the magnitude of both the values of ambient and base temperatures.

Next, a scheme has been established to show the effect of ambient pressure on the optimum design parameters of wet fins as a constraint fin volume. In general, for the design of wet fins, ambient pressure is kept at a constant value. However, depending upon the position of an experimental place with respect to the mean sea level, the ambient pressure may not be a constant. For an example, the ambient pressure at a hilly area is less in comparison with that at sea level. So the design condition at the hilly region is not the same as that at the sea level region. The variation of the fin length, the wet length in a partially wet fin, the base thickness, and the heat transfer rate of an optimum longitudinal fin is investigated as a function of the ambient pressure, which is shown in Fig. 6. In Fig. 6a, the fin length and the wet length in a partially wet surface of an optimum fin are an incremental function with the pressure for a constant relative humidity. On the other hand, both the rate of heat transfer and base thickness decreases with the increase in ambient pressure, as shown in Figs. 6b and 6c. The preceding observation can be explained in the following way: with the increase in ambient pressure for a constant relative humidity, the specific humidity of air decreases and hence condensation of moisture on the fin surface decreases. As the condensation of moisture decreases, both heat transfer rate and base thickness of an optimum fin decrease and, at the same time, length of the optimum fin increases because of a constant volume considered.

The optimization of any fin is done by either maximizing heat transfer rate for a given fin volume or minimizing fin volume for a given heat transfer duty. Thus, depending upon the requirement, any one of the two constraints is used for an optimization study. With this known design parameter, an optimum profile shape is estimated from the solution of the optimality criteria of a fin design along with the constraint condition. The result from the optimization study of wet fins is predicted in Figs. 7–9 as a function of fin volume for longitudinal, annular, and spine, respectively. In comparison, the optimum result for the dry surface condition of each fin is plotted in the corresponding figure. From the figures, it is clear that the optimum parameters, namely, heat transfer rate, fin length, and fin thickness at the base, increase monotonically with the fin volume. Again, it can be demonstrated that the maximum rate of heat transfer is not only a function of the fin volume, but it is also a function of the condition of the surface. A fully wet surface with 100% relative humidity gives a maximum optimum heat transfer rate in comparison with that from any other surfaces. A partially wet surface transfers less amount of heat than that of a fully wet surface fin, and the difference in heat transfer increases gradually with a decrease in the relative humidity. A dry surface fin at an optimum condition transfers the least amount of heat in comparison with the wet surface fin. This observation is expected because of the condensation of moisture on the fin surface for the wet fin and hence latent heat increases the rate of heat transfer. For a lower value of fin volume, the difference in heat transfer among fully wet, partially wet, and dry surfaces is not as important as that for a higher value of fin volume, as shown in Fig. 7a. The length and fin thickness at the base of an optimum fin increases with the fin volume, irrespective of any surface condition. The optimum fin length for fully wet fins is always shorter than that for the

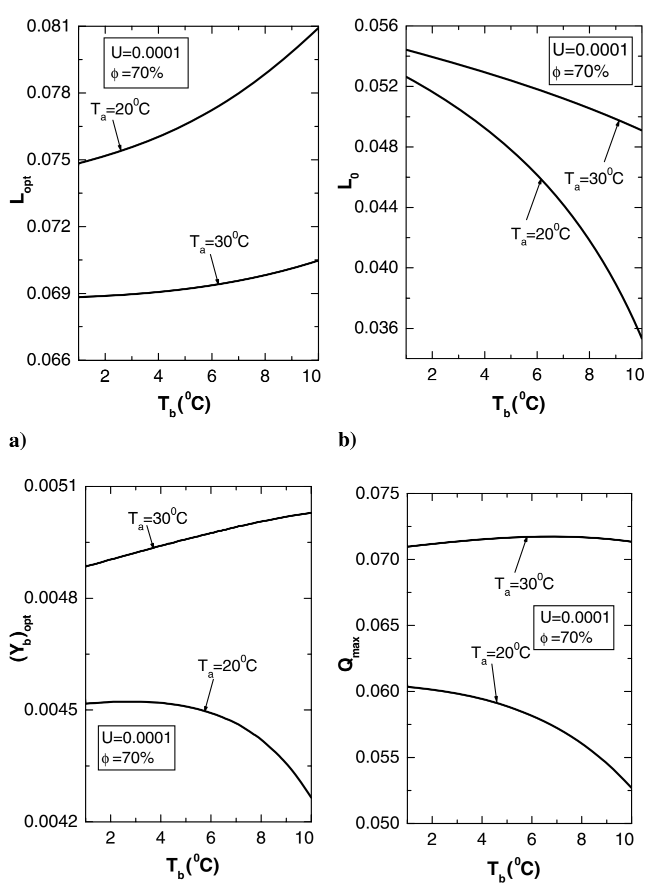

Fig. 5 Effect of base temperature on the design parameters of an optimum longitudinal fin: a) optimum length, b) L_0, c) optimum semifin thickness at the base, and d) maximum heat transfer rate.

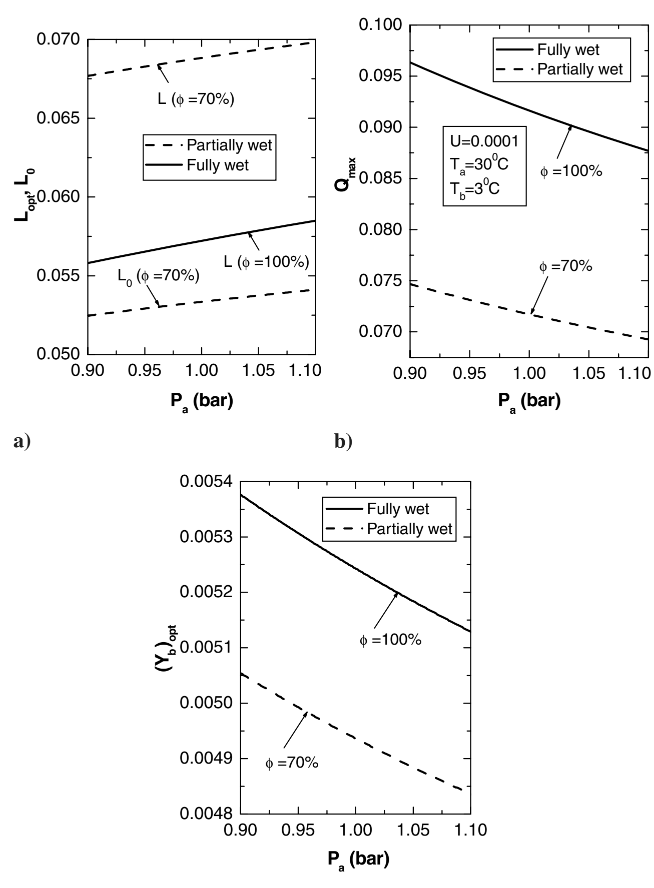

Fig. 6 Effect of ambient pressure on the design parameters of an optimum longitudinal fin: a) optimum length and L_0, b) maximum heat transfer rate, and c) optimum semifin thickness at the base.

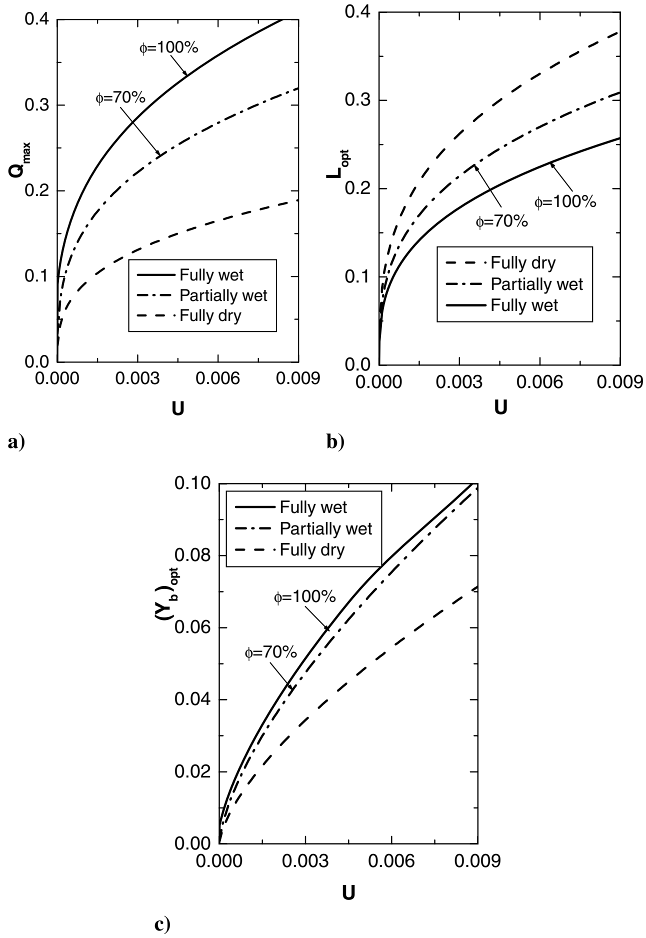

Fig. 7 **Design parameters of an optimum longitudinal fin as a function of fin volume: a) maximum heat transfer rate, b) optimum length, and c) optimum semifin thickness at the base.**

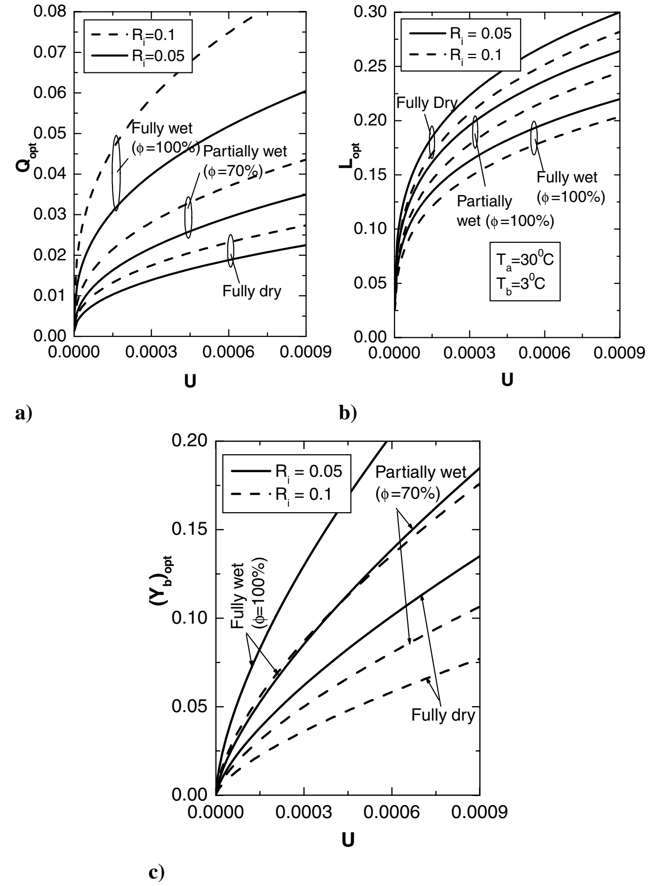

Fig. 8 **Design parameters of an optimum annular fin as a function of fin volume: a) maximum heat transfer rate, b) optimum length, and c) optimum semifin thickness at the base.**

partially wet and dry fins. The optimum length is a maximum for the dry surface condition under the same volume. On the other hand, an opposite trend is noticed for the variation of fin thickness at the base with the fin volume in comparison with the variation of fin length with volume. A similar exercise has been studied for the annular fin and spine, as depicted in Figs. 8 and 9. In the case of an annular fin, the preceding parameters also depend on the thermogeometric parameter R_i. With the increase in R_i, the optimum heat transfer rate increases, and the optimum length and optimum base thickness decreases, separately.

From the preceding optimum results, it is important to mention that the optimum fin shape obtained from an optimization technique with the consideration of only one constraint, either volume or heat transfer rate, is complex in nature and fragile in shape at the tip, as shown in Fig. 2, hence, it may be difficult in the manufacturing process. To overcome this problem and to restrict the length of the fin, fin length is taken as an additional constraint with the fin volume. In this case, the shape of the fin profile and fragile geometry at the tip can be improved significantly. To avoid the same nature of the result, the optimum result under both volume and length constraint is furnished only for the longitudinal fin. The variation of temperature and fin profile is determined under the aforementioned constraint, which is displayed in Fig. 10. From the temperature distribution, it can be mentioned that a dimensionless temperature value at the tip does not vanish and it obviously depends upon the magnitude of the constraints chosen in the optimization process. For a fully wet surface, temperature at the tip is closer to the ambient value in comparison with the partially wet and dry surface conditions. Owing to the condensation of moisture, heat evolved by latent heat of condensation increases the fin surface temperature in the case of a wet fin. With the increase in relative humidity of air, condensation of moisture increases and thus fin surface temperature increases. This observation can be found in Fig. 10a. The profile shape under both volume and length constraints for various surface conditions is illustrated as a function of fin length shown in Fig. 10b. From this figure, it is clear that the profile shape is improved significantly with respect to a profile obtained from only volume constraint, which seems to be suitably compatible in the manufacturing process. However, there is still a difference in shape between dry and wet surface fin profiles under the same design constants adopted.

It is well known that the excess temperature at the tip for the dry surface fin vanishes for only volume constraint to determine an optimum fin shape. In partially wet fins, the preceding condition is also satisfied under the same design constraint. On the other hand, in fully wet fins, tip temperature may be slightly smaller than the ambient value ($\theta_t = -\theta_p$). However, for both the volume and length constraints, the tip temperature depends upon the magnitude of the constraint adopted. The heat transfer rate and tip temperature of a longitudinal fin as a function of fin volume is illustrated in Fig. 11 for an additional constraint $L = 0.05$. For the present design variables, the tip temperature is less than the dew point temperature and, as a result, fin surface is maintained as fully wet. The absolute tip temperature for a dry surface fin is always a smaller value than that for the wet surface fin, as shown in Fig. 11a. Nevertheless, this difference is marginal for higher values of fin volume. Again, with the increase in fin volume, tip temperature of any fin decreases irrespective of its surface conditions. For a higher value of fin volume, tip temperature is closer to the base temperature and thus, under this condition, fin surface may not be functioning properly. After a certain volume, the rate of heat transfer is almost constant with the increase in fin volume. Therefore, attention should be given to the design of fins under an additional length constraint for selecting a fin volume.

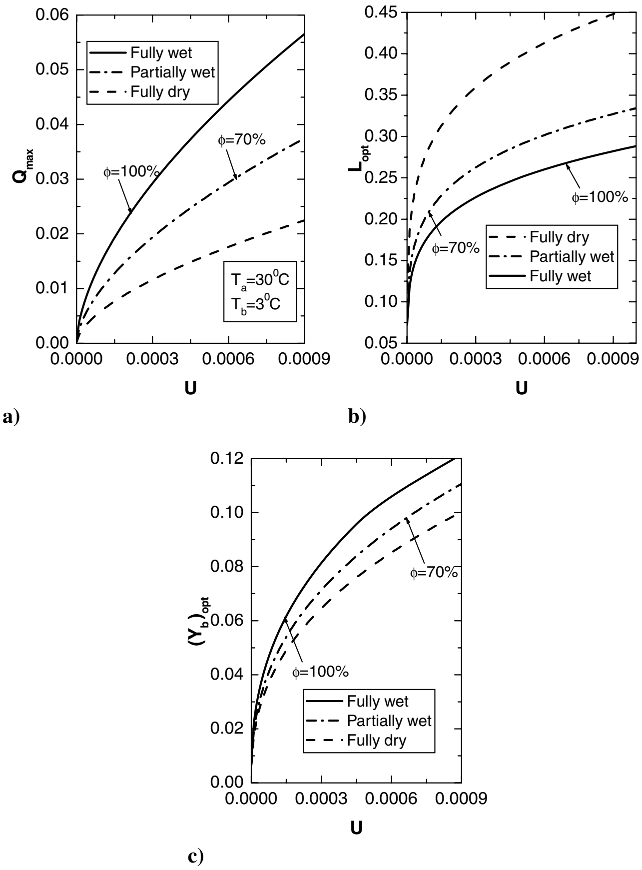

Fig. 9 Design parameters of an optimum pin fin as a function of fin volume: a) maximum heat transfer rate, b) optimum length, and c) optimum fin thickness at the base.

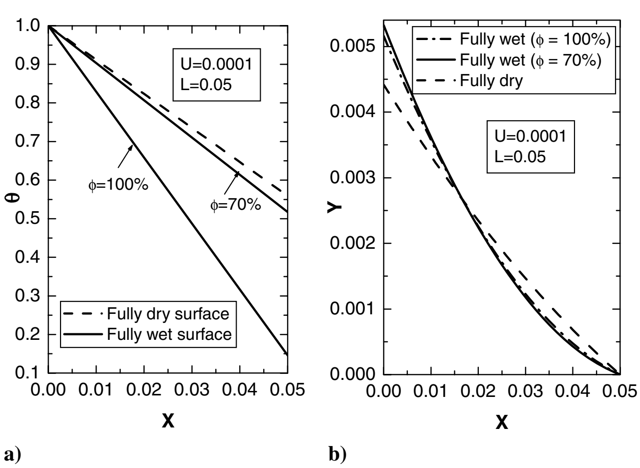

Fig. 10 Variation of temperature and fin profile in a longitudinal fin as a function of length for both volume and length constraints: a) temperature distribution, and b) fin profile.

IV. Conclusions

Depending upon the psychometric conditions of the surrounding air and the requirement of a design condition, the fin surface may be dry, fully wet, or partially wet. A wet surface fin differs from that of a dry surface fin mainly due to the occurrence of mass transfer with the heat transfer mechanism. Thus, the optimum-envelop shape for a wet fin alters with respect to that of dry fins. In the present investigation, the optimum profile shape of different fins, namely, longitudinal, spine, and annular, is determined for dry, fully wet, and partially wet surface conditions using variational principle. The analysis is demonstrated based on a generalized form of heat transfer equations subject to a set of common boundary conditions. For possible requirements of an optimum design, the present analysis has been included for different constraint conditions, namely, fin volume and both fin length and fin volume. From the results, it is clear that the optimum design variable for wet fins is not only dependent upon the design constraints but is also a function of psychometric properties of air. Unlike dry and partially wet surface fins, tip temperature for the fully wet optimum fins under the volume constraint may be less than the surrounding temperature. The optimum profile for a wet surface fin differs clearly from that of dry surface fins, and this difference not only depends upon the design constraint adopted but also depends upon the psychometric parameters involved with this design process. A significant change in optimum design variables has been noticed with the change in design constants, such as relative humidity, dry bulb temperature, base temperature, and ambient pressure. Extra caution is required for the selection of the magnitude of constraints for the design of fins under both volume and length constraints. Finally, because the present work forwards a unified analysis of optimum fin design, it may be helpful to the designer to select a particular type of fin once the primary heat transfer surface and volume, or primary heat transfer surface and both volume and length, of a fin is specified.

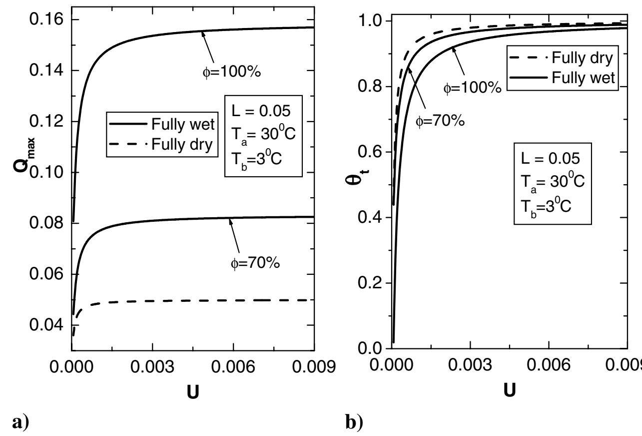

Fig. 11 Variation of maximum heat transfer rate and tip temperature as a function of fin volume for a longitudinal fin for both volume and length constraints: a) maximum heat transfer rate, and b) dimensionless tip temperature.

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